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THE INFLUENCE OF WEAK ATMOSPHERIC INHOMOGENEITIES UPON THE PROPAGATION OF SOUND AND LIGHT

by A. M. Obukhov

Translated by W. C. Hoffman

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SUMMARY

The problem of wave propagation in a medium with fluctuating refractive index is considered, this problem being of interest from the standpoint of the theory of twinkling of stars and atmospheric acoustics. The problem is reduced to linearised equations satisfied by the phase and logarithmic amplitude of the propagating wave. Limiting cases are investigated and particular examples are considered to indicate the limits of validity of the geometric optics treatment and the nature of the corrections required to take account of diffraction effects.

Small atmospheric inhomogenieties arising from turbulence exert a profound influence on propagation of sound and light, causing the characteristic fluctuations of the associated field. V. A. Krasilnikov [1, 2] has investigated experimentally phase and amplitude fluctuations of sound waves propagating through the atmosphere. Similar fluctuations of light waves are exhibited, for instance, by the phenomenon of twinkling of stars, which are evidently closely connected with turbulent fluctuations of atmospheric temperature. [3, 4]*

References 1, 3 and 8 contain some computations of the phase and amplitude fluctuations of a wave propagating in a turbulent medium for oblique incidence. However, this work is based on the geometric optics (acoustics) approximation, which evidently in certain cases is inadequate and gives rise to a discrepancy between computed and experimental values. For instance in the geometric optics approximation the amplitude fluctuations are proportional to the 3/2 power of the (turbulent) layer thickness, whereas observations generally indicate a considerably lower order of growth.

In the present paper an attempt is made to consider the problem of amplitude and phase fluctuations of a scalar wave field on a more general basis and by means of more accurate approximations taking into account diffraction effects, to thus clarify the limits of validity of the geometric optics approximation.

1. Derivation of the Basic Equations.

Let us consider a scalar wave field given by $\varphi(x, y, z, t) = \varphi(r,t)$ which satisfies the equation

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi, \qquad (1)$$

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*A theoretical analysis of turbulent fluctuations of the atmospheric temperature and certain related experimental data can be found in references [5 - 7].

where the propagation-velocity c of the wave is a function of position.*
The time dependence of c will be neglected, assuming that the characteristics of the medium are slowly-varying in time (i.e., with respect to the carrier frequency). Further it will be supposed that the deviations of c from some mean value c_o:

$$c = \frac{c_o}{n}, \tag{2}$$

are small everywhere in the medium and for all time. The refractive index n fluctuates around unity

$$n = 1 + \mu, \quad (\mu \ll 1) \tag{2'}$$

(in this case it is possible to speak of a slightly inhomogeneous medium).

Let us suppose that the half-space $x < 0$ is homogeneous (i.e., $n = 1$ for $x < 0$) and that plane waves are incident at the plane $x = 0$ from the direction of the negative x-axis. The problem consists of determining the characteristics of the wave field in the plane of observation $x = L$. It is clear that this problem is closely related to the problem of diffraction of light by ultrasonic waves, which has been investigated by S. M. Rytov [2], but however differs in that we cannot assume periodic variation of the refractive index.

Consider the case when $\varphi(\underset{\sim}{r}, t)$ is a monochromatic wave of frequency ω:

$$\varphi(\underset{\sim}{r}, t) = A(\underset{\sim}{r}) \exp\left\{i\left[\omega t - S(\underset{\sim}{r})\right]\right\}, \tag{3}$$

*For the optical problem (e.g., twinkling of stars) the scalar equations apply if polarisation effects may be neglected.

with amplitude $A(\underset{\sim}{r})$ and phase $S(\underset{\sim}{r})$. It is appropriate to introduce a complex function

$$\psi(r) = S + i \ln \frac{A}{A_o}, \tag{4}$$

so that

$$\varphi(\underset{\sim}{r}, t) = A_o \exp\left\{i\left[\omega t - \psi(\underset{\sim}{r})\right]\right\}, \tag{5}$$

where A_o is the amplitude of the (unperturbed) incident wave. We have thus passed from equation (1) to the eikonal equation (sic)

$$(\nabla\psi)^2 + i\Delta\psi = k_o^2 n_1^2 \qquad k_o = \frac{\omega}{c_o} \tag{6}$$

In the case of a homogeneous medium we would have in place of equation (6) the equation

$$(\nabla\psi_o)^2 + i\Delta\psi_o = k_o^2 \tag{6'}$$

(ψ_o is the equivalent of ψ for a homogeneous medium). Subtracting (6′) from (6) yields

$$2(\nabla\psi_o \cdot \nabla\psi') + i\Delta\psi' = 2\mu k_o^2 + \left[\mu^2 k_o^2 - (\nabla\psi')^2\right],$$

where

$$\psi' = \psi - \psi_o, \qquad \nabla\psi' = \nabla\psi - \nabla\psi_o.$$

Having assumed that $\nabla\psi'$ is of the order of μ, one has a basis for neglecting terms of order μ^2 in the square brackets on the right. We then obtain a linear equation in ψ'

$$2\nabla\psi'\cdot\nabla\psi_0 + i\,\Delta\psi' = 2\mu k_0^2, \tag{7}$$

which will serve as a basis for the further analysis. Similar linearised eikonal equations were employed for the first time in the work of S. M. Rytov[9] in 1937.

We note that $\mathrm{Re}\,\psi' = S - S_0$ is the fluctuating (perturbed) phase of the wave brought about by fluctuations of the refractive index n, and $\mathrm{Im}\,\psi' = \ln (A/A_0)$ is the logarithm of the ratio of the amplitudes of perturbed and incident fields. Because in the derivation of equation (7) we have supposed only that $\nabla\psi'/k_0 \ll 1$, it is clear that no requirement of smallness of phase and amplitude perturbations (with respect to π and A_0, respectively) has been imposed. What is required is that the change of phase and logarithmic amplitude over one wavelength (in an arbitrary direction) be sufficiently small.* This condition is satisfied for instance by the refraction of the primary ray at small angles or by diffuse radiation provided the energy of the diffuse component is small in comparison with the energy of the primary field.

If one substitutes in (7) the function $\psi_0 = k_0 x$, corresponding to a plane wave propagating in the direction of the positive x-axis, there results the equation

$$2k_0\frac{\partial\psi'}{\partial x} + i\,\Delta\psi' = 2\mu k_0^2 \tag{8}$$

From this equation it is possible to obtain the geometric optics

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*We note that the classical method of solving the wave equation (1) by means of a method of perturbation theory, which is often applied in scattering problems, becomes inapplicable at large L because of "phase piling up" to values comparable with π (cf. [10]).

approximation in the way used by V. A. Krasilnikov[8]. Let us introduce the optical path length $\theta' = S'/k_0$ and se[para]te real and imaginary parts of (8), having previously divided the whole equation through by $2k_0^2$.

$$\frac{\partial \theta'}{\partial x} - \frac{1}{2k_0} \Delta \left(\ln \frac{A}{A_0} \right) = \mu, \tag{9}$$

$$\frac{\partial \ln (A/A_0)}{\partial x} + \tfrac{1}{2} \Delta \theta' = 0. \tag{10}$$

Assuming that for $k_0 \longrightarrow \infty$ the quantities θ' and $\ln (A/A_0)$ tend toward finite limits, one obtains in the limit the equations of V. A. Krasilnikov:

$$\frac{\partial \theta'}{\partial x} = \mu,$$

that is

$$\theta' = \int_0^x \mu \, dx \tag{11}$$

$$\ln \frac{A}{A_0} = -\tfrac{1}{2} \int_0^x \Delta \theta' \, dx \tag{12}$$

or, substituting (11) into (12),

$$\left[\ln \frac{A}{A_0} \right]_{x=L} = \tfrac{1}{2} \left[\mu(0) - \mu(L) \right] - \tfrac{1}{2} \int_0^L (L-x) \left(\frac{\partial^2 \mu}{\partial y^2} + \frac{\partial^2 \mu}{\partial z^2} \right) dx$$

$$\approx -\tfrac{1}{2} \int_0^L (L-x) \, \Delta_2 \mu \, dx, \tag{13}$$

where $\Delta_\perp = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the "transverse" Laplace operator. Retaining the diffraction term $-\frac{1}{2k_0}\Delta(\ln A/A_0)$ in equation (9) makes possible more general results by approximating the wave effects and thus permits one to obtain estimates of the limits of Krasilnikov's theory.

A solution to equation (8) will be sought of the form

$$\Psi' = \exp(ik_0 x)\, w. \tag{14}$$

Substitution of (14) into (8) shows that w satisfies the inhomogeneous wave equation

$$\Delta w + k_0^2 w = -f(x,y,z) \tag{15}$$

where

$$f = 2i\mu k_0^2 \exp(-ik_0 x).$$

It is well known that the solution to equation (15) can be written in the form*

$$w(P) = \frac{1}{4\pi}\iiint_T \frac{e^{(-ik_0 r)}}{r} f(M)\, dV_M, \tag{16}$$

where T is the domain of integration and

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

is the distance between the point of observation P(x,y,z) and the variable point M(ξ,η,ζ). If we now pass back to the function Ψ', we find

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*The question of what conditions to impose on w to exclude singular solutions will not detain us here. In concrete examples it is not difficult to verify that other solutions (which contain terms of "advanced potential" type) lead to physically absurd expressions for Ψ'

$$\Psi'(x,y,z) = \frac{ik_0^2}{2\pi} \iiint_T \frac{\exp\{-ik_0[r-(x-\xi)]\}}{r} \mu(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta \tag{17}$$

Thus we are successful in obtaining an exact solution to equation (8), the source function being of the form

$$-\frac{1}{4\pi} \frac{\exp\{-ik_0[r-(x-\xi)]\}}{r}$$

For the further investigation we restrict ourselves to the case where the wavelength, though finite, is several times smaller than the characteristic dimensions of the perturbation, that is, to Fresnel diffraction. In this case one naturally assumes that the field at the point (L, 0, 0) results only from the contributions of the medium inside a narrow cone with vertex angle $\varepsilon << 1$ and vertex at the source; in other words it is possible to neglect the scattering at large angles (in particular, the backward scattering).*

Setting

$$r^2 = (x-\xi)^2 + \rho^2, \qquad \rho = \sqrt{(y-\eta)^2 + (z-\zeta)^2},$$

one sees that our assumptions mean that the essential contribution to the integral (17) comes from that region for which

$$x-\xi \geq 0 \qquad \text{and} \qquad \rho << x-\xi,$$

i.e., $\rho/x-\xi << 1$

It then follows that

$$r \approx (x-\xi) + \frac{\rho^2}{2(x-\xi)} \tag{18}$$

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*Admissibility of values of ε can be decided by means of the scattering indicatrix. For the case $L/\lambda > 1$ and weak perturbations this can be computed by the method of K. S. Shifrin.

If one replaces $1/r$ in the integrand of (17) by $\frac{1}{x-\xi}$ and $r-(x-\xi)$ by the second order terms of equation (18), then in the region of important contributions to the field at (L, 0, 0) one has approximately

$$\frac{1}{r} \exp\left\{-ik_o\left[r-(x-\xi)\right]\right\} \approx \frac{1}{x-\xi} \exp\left[-\frac{ik_o\rho^2}{2(x-\xi)}\right] \tag{19}$$

The expression for $\psi'(x,y,z)$ can then be written approximately as

$$\psi'(x,y,z) = \frac{ik_o^2}{2\pi} \iiint_{T'} \frac{\exp\left[-\frac{ik_o\rho^2}{2(x-\xi)}\right]}{x-\xi} \mu(\xi,\eta,\zeta)\, d\eta\, d\zeta\, d\xi, \tag{20}$$

where T' is the region such that $x-\xi \geq 0$ (i.e., situated between the source of the wave and the point of observation).* We remark that formula (20) constitutes the exact solution to the equation

$$2k_o\frac{\partial\psi'}{\partial x} + i\left(\frac{\partial^2\psi'}{\partial y^2} + \frac{\partial^2\psi'}{\partial z^2}\right) = 2\mu k_o,$$

which differs from (8) only in that the "transverse" Laplace operator $\Delta_\perp\psi'$ appears instead of $\Delta\psi'$ (i.e., the term in $\frac{\partial^2\psi'}{\partial x^2}$ is omitted). The effect of neglecting the term $\frac{\partial^2\psi'}{\partial x^2}$ has an effect on the solution similar to that for the purely periodic perturbation justified by S. M. Rytov [9].

As a means of comparing the results given by the formula (20) with those following from the geometric optics approximation we compute the phase and amplitude perturbations at the point (L, 0, 0) for the following increment to the refractive index:

$$\mu = \exp\left(-\frac{y^2+z^2}{2\ell^2}\right)\mu_1(x) \tag{21}$$

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*We can now extend formally the integration over the whole space T', not only in the interior of the control cone, because outside this cone the source function (both approximate and precise) rapidly oscillates, so that integration over the space outside the cone yields negligible contribution for sufficiently smooth variation of $\mu(x,y,z)$.

(The scale ℓ evidently characterizes the transverse dimension of the perturbation.)

Performing the integration in η and ζ in (20) yields

$$\Psi(L,0,0) = \int_0^L \frac{k_0 + i(L-\xi)/\ell^2}{1 + (L-\xi)^2/k_0^2\ell^4}\,\mu_1(\xi)\,d\xi.$$

In view of (21) this expression can be put in the form:

$$\Psi(L,0,0) = \int_0^L \frac{k_0}{1 + (L-\xi)^2/(k_0^2\ell^4)}\,\mu(\xi,0,0)\,d\xi - \qquad (22)$$
$$-\frac{i}{2}\int_0^L \frac{L-\xi}{1 + (L-\xi)^2/(k_0^2\ell^4)}\,\Delta_1\mu(\xi,0,0)\,d\xi.$$

The phase fluctuations are given by the first summand on the right-hand side of this expression and the fluctuations of the logarithmic amplitude by the second summand. This result differs from formulas (11) and (13) obtained under the geometric optics approximation by the presence in the integrand of the factor $1/\left[1 + (L-\xi)^2/k_0^2\ell^4)\right]$. It follows that the geometric optics formulas give the correct result for the case $L/(k_0\ell^2) \ll 1$. Introducing the wavelength $\lambda = 2\pi/k_0$ enables one to write the criterion just given in the form

$$\sqrt{L\lambda} \ll \sqrt{2\pi}\,\ell\ .$$

This means that the first Fresnel zone ought to be considerably less than the transverse dimension of the perturbation. For a given wavelength λ the geometric optics approximate formulas hold only for distances considerably less than the critical length $L_{cr} = 2\pi\ell^2/\lambda$. The absolute magnitude of phase and amplitude fluctuations are, as one expects, less affected in theory by diffraction effects than in the geometric optics approximation. We shall return to this matter after a statistical treatment of the problem.

2. Statistical Treatment of the Problem.

Up till now we have supposed that the values of the refractive index $n(x,y,z)$ constitute a known function of coordinates. In fact, however, all characteristics of the medium (including the refractive index n) are constantly undergoing chaotic changes in time and space, so that a proper definition of $n(x,y,z,t)$ can only be made on a statistical basis. The role of the "impossible" value $n = 1$ will be that of the mean value of the refractive index, i.e.,

$$\overline{n(x,y,z)} = 1, \qquad \mu = n - \bar{n}.$$

In view of the statistical character of the refractive index fluctuations $\mu(x,y,z) = \mu(M)$ we introduce the corresponding covariance function

$$B(M_1, M_2) = \overline{\mu(M_1)\mu(M_2)} \; . \tag{23}$$

In the sequel it will be assumed that $\mu(M)$ is a stationary process so that $B(M_1, M_2)$ can be written as $B(r)$, where r is the distance between the points M_1 and M_2.

A special notation for the real and imaginary parts of the kernel of the integral (20) will now be introduced:

$$\Phi_1(a,\rho) = \frac{1}{2\pi a}\sin\frac{\rho^2}{2a}, \qquad \Phi_2(a,\rho) = \frac{1}{2\pi a}\cos\frac{\rho^2}{2a}$$

Then the integral (20) can be written as

$$S'(L,0,0) = \mathrm{Re}\,\Psi' = k_0\int_0^L\int_{-\infty}^{\infty}\int \Phi_1\left(\frac{L-\xi}{k_0},\rho\right)\mu(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta, \tag{24}$$

$$\ln\frac{A}{A_0}\bigg|_{(L,0,0)} = \mathrm{Im}\,\Psi' = k_0\int_0^L\int_{-\infty}^{\infty}\int \Phi_2\left(\frac{L-\xi}{k_0},\rho\right)\mu(\xi,\eta,\zeta)\,d\xi\,d\eta\,d\zeta. \tag{25}$$

Squaring right and left sides of (24) and (25), averaging and replacing the lengths L, ρ and so on by the dimensionless values $\bar{L} = k_0 L$, $\bar{\rho} = k_0 \rho$ and so on (i.e., $\lambda/2\pi$ is unit for all lengths), we obtain the following expressions for the statistical parameters of the phase and amplitude fluctuations:

$$\overline{S'(L,0,0)^2} = \int_0^L \int_0^L \iiiint_{-\infty}^{\infty} \Phi_1(\bar{L}-\bar{\xi}_1,\bar{\rho}_1)\Phi_1(\bar{L}-\bar{\xi}_2,\bar{\rho}_2) B(r)\, d\bar{\xi}_1 d\bar{\xi}_2 d\bar{\eta}_1 d\bar{\eta}_2 d\bar{\zeta}_1 d\bar{\zeta}_2 \quad (26)$$

$$\overline{\left[\ln \frac{A}{A_0}\right]^2_{(L,0,0)}} = \int_0^L \int_0^L \iiiint_{-\infty}^{\infty} \Phi_2(\bar{L}-\bar{\xi}_1,\bar{\rho}_1)\Phi_2(\bar{L}-\bar{\xi}_2,\bar{\rho}_2) B(r)\, d\bar{\xi}_1 d\bar{\xi}_2 d\bar{\eta}_1 d\bar{\eta}_2 d\bar{\zeta}_1 d\bar{\zeta}_2 \quad (27)$$

where

$$\bar{\rho}_1^2 = \bar{\eta}_1^2 + \bar{\zeta}_1^2, \qquad \bar{\rho}_2^2 = \bar{\eta}_2^2 + \bar{\zeta}_2^2,$$

$$r^2 = r'^2 k_0^2 = \frac{1}{k_0^2}\left[(\bar{\xi}_1-\bar{\xi}_2)^2 + (\bar{\eta}_1-\bar{\eta}_2)^2 + (\bar{\zeta}_1-\bar{\zeta}_2)^2\right].$$

The change of variables $y = 1/2\,(\bar{\eta}_1 + \bar{\eta}_2)$; $\eta = \bar{\eta}_1 - \bar{\eta}_2$; $z = 1/2\,(\bar{\zeta}_1 + \bar{\zeta}_2)$; $\zeta = \bar{\zeta}_1 - \bar{\zeta}_2$ makes it possible to perform the integration in y and z. For that purpose we may employ the familiar expression from probability theory

$$\frac{1}{4\pi^2 \sigma_1 \sigma_2} \iint_{-\infty}^{\infty} \exp\left[-\frac{(y_1+y)^2+(z_1+z)^2}{2\sigma_1}\right] \exp\left[-\frac{(y_2-y)^2+(z_2-z)^2}{2\sigma_2}\right] dy\, dz =$$

$$= \frac{1}{2\pi(\sigma_1+\sigma_2)} \exp\left[-\frac{(y_1+y_2)^2+(z_1+z_2)^2}{2(\sigma_1+\sigma_2)}\right]$$

("The result of convoluting two Gaussian laws is again a Gaussian law.")

Substituting first in this expression $y_1 = y_2 = (1/2)\eta$; $z_1 = z_2 = 1/2\,\zeta$;

$S_1 = i\alpha_1$; $S_2 = -i\alpha_2$; and then $y_1 = y_2 = (1/2)\eta$; $z_1 = z_2 = 1/2\,\zeta$; $S_1 = i\alpha_1$; $S_2 = i\alpha_2$: adding and subtracting the resulting two equations and splitting off in both cases the real parts, one is led to the expressions

$$\int\int \Phi_1\left(a_1; \sqrt{\left(y+\frac{\eta}{2}\right)^2+\left(z+\frac{\zeta}{2}\right)^2}\right)\Phi_1\left(a_2; \sqrt{\left(y-\frac{\eta}{2}\right)^2+\left(z-\frac{\zeta}{2}\right)^2}\right)dy\,dz = \tag{28}$$

$$= \frac{1}{2}\left\{\Phi_1(a_1-a_2; \rho) + \Phi_1(a_1+a_2; \rho)\right\},$$

$$\int\int \Phi_2\left(a_1, \sqrt{\left(y+\frac{\eta}{2}\right)^2+\left(z+\frac{\zeta}{2}\right)^2}\right)\Phi_2\left(a_2, \sqrt{\left(y-\frac{\eta}{2}\right)^2+\left(z-\frac{\zeta}{2}\right)^2}\right)dy\,dz = \tag{29}$$

$$= \frac{1}{2}\left\{\Phi_1(a_1-a_2, \rho) - \Phi_1(a_1+a_2; \rho)\right\},$$

where $\rho^2 = \zeta^2 + \eta^2$.

It thus follows from (26) and (27) that

$$\overline{S'(L,0,0)^2} = \frac{1}{2}\left(I_1 + I_2\right), \tag{30}$$

$$\overline{\left[\ln\frac{A(L,0,0)}{A_0}\right]^2} = \frac{1}{2}\left(I_1 - I_2\right), \tag{31}$$

where

$$I_1 = \int_0^{\bar L}\int_0^{\bar L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_1(\bar\xi_1 - \bar\xi_2, \rho)\,B(r)\,d\eta\,d\zeta\,d\bar\xi_1\,d\bar\xi_2, \tag{32}$$

$$I_2 = \int_0^{\bar L}\int_0^{\bar L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_1\left(2\bar L-(\bar\xi_1+\bar\xi_2), \rho\right)B(r)\,d\eta\,d\zeta\,d\bar\xi_1\,d\bar\xi_2, \tag{33}$$

$$r^2 = \frac{1}{k_0^2}\left[(\bar\xi_1 - \bar\xi_2)^2 + \rho^2\right].$$

The formulas obtained seem to be rather general and can be used to compute phase and amplitude fluctuations for correlation functions $B(r)$ of arbitrary form. As an example let us consider the following correlation function:

$$B(r) = \overline{\mu(M_1)\mu(M_2)} = B_0 \exp\left(-\frac{r^2}{2\ell_1^2}\right), \tag{34}$$

where $B_0 = \overline{\mu^2}$ and r is the distance between the points M_1 and M_2.

One may think that asymptotic laws which are appropriate to such a correlation function hold also (with precision to values of numerical coefficients) for any other sufficiently smooth correlation function $B(r)$ which decreases sufficiently rapidly at infinity.

Substituting the function (34) into formulas (32) and (33) enables one to explicitly perform the integration in η and ζ.

Returning now to the previous dimensional quantities, we find

$$I_1 = B_0 k^2 \int_0^L \int_0^L \frac{1}{1 + \frac{(\xi_1 - \xi_2)^2}{k_0^2 \ell_1^4}} \exp\left[-\frac{1}{2\ell_1^2}(\xi_1 - \xi_2)^2\right] d\xi_1\, d\xi_2 \tag{35}$$

$$I_2 = B_0 k^2 \int_0^L \int_0^L \frac{1}{1 + \frac{[2L - (\xi_1 + \xi_2)]^2}{k_0^2 \ell_1^4}} \exp\left[-\frac{1}{2\ell_1^2}(\xi_1 - \xi_2)^2\right] d\xi_1\, d\xi_2 \tag{36}$$

One observes that the factor $\exp\left[-\frac{1}{2\ell_1^2}(\xi_1 - \xi_2)^2\right]$ inside the integral represents the correlation function of the fluctuating index of refraction along the x-axis (i.e., along the direction of propagation of the incident wave).

For $L \gg \ell_1$ the double integral appearing in (35) and (36) can be reduced to a single integral by using the fact that the function differs

sensibly from zero onl in a narrow strip about the line $\xi_1 = \xi_2$. The corresponding asymptotic representation is easily obtained by a change to the variables $x = 1/2(\xi_1 + \xi_2)$ and $\xi = \xi_1 - \xi_2$. If $L >> \ell_1$, the ξ- integration can be extended to the whole real axis with negligible error. If it even happens that $k_o\ell_1 >> 1$, then it is possible to neglect the summand $(\xi_1 - \xi_2)^2/k_o^2\ell_1^4$ with respect to unity [because of the exponential factor the contribution of the denominator for $(\xi_1 - \xi_2) >> \ell_1$ is small in general]. Hence there easily follows the following asymptotic representations of the integrals I_1 and I_2 for $L >> \ell$ and $k_o\ell_1 >> 1$ (i.e., $\ell_1 >> \lambda/(2\pi)$):

$$I_1 \approx \sqrt{2\pi}\, B_o\, \ell_1 k_o^2 L, \tag{37}$$

$$I_2 \approx \sqrt{\frac{\pi}{2}}\, B_o\, \ell_1^3 k_o^3 \arctan\left(\frac{2L}{k_o \ell_1^2}\right) \tag{38}$$

We remark that we introduce the dimensionless parameter

$$D = \frac{2L}{k_o\ell_1^2}, \tag{39}$$

where it has naturally been assumed that $L >> \ell$ and $\ell_1 >> \lambda$, and that it can in general be allowed to take arbitrary values in computing the asymptotic representations of I_1 and I_2.

From (30), (31), (37), and (38) follows the asymptotic expressions for the mean square fluctuations of phase and logarithmic amplitude:

$$\overline{S^2} \approx \sqrt{\frac{\pi}{2}}\, B_o\, \ell_1 k_o^2 L \left(1 + \frac{1}{D}\arctan D\right), \tag{40}$$

$$\overline{\left(\ln\frac{A}{A_o}\right)^2} \approx \sqrt{\frac{\pi}{2}}\, B_o\, \ell_1 k_o^2 L \left(1 - \frac{1}{D}\arctan D\right). \tag{41}$$

Instead of the conventional "radius of correlation" l_1 it is possible to introduce the "integrated scale of the perturbation," i.e., the integral of the correlation function along the ray

$$\bar{l} = \frac{1}{B_0}\int_0^\infty B(\xi)\,d\xi = \int_0^\infty \exp\left(-\frac{\xi^2}{2l_1^2}\right)d\xi = \sqrt{\frac{\pi}{2}}\,l_1. \qquad (42)$$

Replacing besides the wave number k_0 by the wavelength $\lambda = 2\pi/k_0$, one can write the parameter as

$$D = \frac{L\lambda}{2\bar{l}^2}, \qquad (43)$$

and the formulas become

$$\overline{S^2} \approx (2\pi)^2 B_0 \frac{\bar{l}L}{\lambda^2}\left(1+\frac{1}{D}\arctan D\right), \qquad (44)$$

$$\overline{\left(\ln\frac{A}{A_0}\right)^2} \approx (2\pi)^2 B_0 \frac{\bar{l}L}{\lambda^2}\left(1-\frac{1}{D}\arctan D\right) \qquad (45)$$

We recall that here $B_0 = \overline{\mu^2} = (n-\bar{n})^2$ is the mean square of the refractive index fluctuations.

Let us now consider the limiting cases. For small values of the parameter D (for $D \ll 1$)

$$\arctan D \quad D, \qquad 1-\frac{1}{D}\arctan D \approx \frac{1}{3}D^2$$

and the corresponding mean square values are

$$\overline{S'^2} \approx (2\pi)^2\, 2B_0 \frac{L\bar{l}}{\lambda^2}, \qquad \overline{\left(\ln\frac{A}{A_0}\right)^2} \approx \frac{\pi^2}{3} B_0 \frac{L^3}{\bar{l}^3}$$

$$\sigma_s = \frac{2\sqrt{2}\,\pi}{\lambda}\sqrt{\overline{\mu^2}}\sqrt{L\bar{l}}, \qquad (46)$$

$$\sigma_{\ln(A/A_0)} \approx \frac{\pi}{\sqrt{3}}\, \sigma_\mu \left(\frac{L}{\bar{l}}\right)^{3/2} \tag{47}$$

($\sigma_{S'}$, $\sigma_{\ln(A/A_0)}$, σ_μ are mean square values of the respective quantities). As one expects, these are well known formulas of geometric optics (expressions (46) and (47) agree with those obtained by V. A. Krasilnikov [8]). The case $D \ll 1$ corresponds to comparatively small values of the distance L, viz., $L \ll L_{cr}$, where $L_{cr} = \overline{l^2}/(\lambda/2)$.

For $D \gg 1$ (i.e., for $L \gg L_{cr}$) we have $\arctan D \approx \frac{\pi}{2}$, and formulas (40) and (41) give

$$\overline{S'^2} \approx (2\pi)^2 B_0 \frac{L\bar{l}}{\lambda^2}, \qquad \sigma_{S'} \approx \frac{2\pi}{\lambda}\sigma_\mu \sqrt{L\bar{l}}, \tag{48}$$

$$\overline{\left(\ln\frac{A}{A_0}\right)^2} \approx (2\pi)^2 B_0 \frac{L\bar{l}}{\lambda^2}, \qquad \sigma_{\ln(A/A_0)} \approx \frac{2\pi}{\lambda}\sigma_\mu \sqrt{L\bar{l}}. \tag{49}$$

Thus for large values of L the formula for the phase fluctuations simplifies to the corresponding formula of the geometric optics approximation, differing from the latter only in the numerical factor $1/\sqrt{2}$ (in particular the proportionality of $\sigma_{S'}$ to the square root of L is preserved). At the same time the dependence of the amplitude fluctuations on L changes sharply upon crossing the value L_{cr}: in the same fashion as with small L-values the parameter $\sigma_{\ln(A/A_0)}$ increases proportionally to $L^{3/2}$ and is independent of wavelength (in agreement with the theory of geometric optics). For larger values of L the fluctuations of the logarithmic amplitude follow the same law as the phase fluctuations: $\sigma_{\ln(A/A_0)}$ is proportional to $\sqrt{L}$ and in addition begins to depend on wavelength.

We note moreover that, according to the above formulas, in every case

$\sigma_{\ln(A/A_0)} < \sigma_S$ where

$$\frac{\sigma_{\ln(A/A_0)}}{\sigma_S} = \sqrt{\frac{D - \arctan D}{D + \arctan D}}$$

The dependence on wavelength (or frequency) given by the theory can be used to explain the phenomenon of color changes in twinkling stars.

In conclusion we shall estimate the order of magnitude of L_{cr} for twinkling of stars. Assume that the light wavelength λ is 5×10^{-5} cm. and that $\bar{l}$ (the "internal scale" of the atmospheric turbulence) is 2 cm. For these values $L_{cr} = 1.6 \times 10^5$ cm. = 1.6 km. The value of the scale so obtained compares with the thickness of that layer of the troposphere where one expects the great majority of the temperature fluctuations and consequently the fluctuations of refractive index. Thus even in the problem of twinkling of stars where it would appear that one should find a very favorable situation for using geometric optics methods (since $\lambda << l_1$) due to the great thickness of the layer one encounters diffraction effects (in the sense of Fresnel).

Qualitative confirmation of the results of the present investigation can be gleaned from experimental data on amplitude fluctuations of sound and light waves in the atmosphere. Thus measurements of the sound amplitude carried out by Krasilnikov [2] show that the fluctuations of phase and logarithmic amplitude are of the same order of magnitude but that the latter are somewhat less. The growth of the logarithmic amplitude fluctuations with increasing frequency is also observed. We also observe in Krasilnikov's results [2] the $\sqrt{L}$-growth of the logarithmic amplitude fluctuations of the sound wave for larger L (but not the law $L^{3/2}$; this latter law of growth sharply contradicts all observations). An analogous deduction can be drawn

from Siedentopf's [11] measurements of the intensity fluctuations of light received from distant sources.

Only qualitative deductions from the proposed theory can be considered here. For a quantitative comparison based on experimental results it is necessary first of all to carry out calculations for whatever particular correlation function is characteristic of the fluctuations of the atmospheric refractive index. The converse problem is also of considerable interest: To obtain estimates for the characteristics of atmospheric turbulence (intensity, scale) from statistical analysis of observed light intensity fluctuations (twinkling of stars) or the sound field characteristic of atmospheric turbulence. Consideration of these problems falls outside the scope of the present paper, however.

USSR Academy of Science
Geophysical Institute

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